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Some Decomposition of Normal Projective Curvature Tensor II

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Abstract: **In the present paper, a Finsler space F_n for which the normal projective curvature tensor N^i_{jkh} satisfies $N^i_{jkh} = y^i Y_{jkh}$, where Y_{jkh} is non-zero tensor field called *decomposition tensor field*, we discuss decompose the normal projective curvature tensor in recurrent Finsler space.**

Keywords: **a Finsler space, decompositions tensor in recurrent Finsler space.**

1. INTRODUCTION

K. Takano [8] discussed recurrent affine motion in a recurrent non-Riemannian space and later on he discussed a recurrent whose curvature tensor is decomposable. Ram Hit [6], B. B. Sinha and S. P. Singh [7], H. D. Pande and H. S. Shakla [2] studied a recurrent Finsler space whose curvature tensor is decomposable. H. D. Pande and H. S. Shakla [2] considered a recurrent Finsler space whose curvature tensor is decomposable. P. N. Pandey [3], H. D. Pande and T. A. Khan [1], Prateek Mishra, Kaushal Srivastava, S. B. Mishra [5] dialed with the problem decomposability of curvature tensor, a necessary and sufficient condition for decomposability of curvature tensor has been obtained.

Let us consider a set of quantities g_{ij} defined by

(1.1) $g_{ij}(x,y) = \frac{1}{2}\dot\partial_i \dot\partial_j F^2(x,y).$

The tensor $g_{ij}(x,y)$ is positively homogeneous of degree zero in y^i and symmetric in i and j. According to Euler's theorem on homogeneous functions, the vector y_i satisfies the following relation

(1.2) $y_i y^i = F^2,$

Berwald covariant derivative of the metric function F and vector y^i vanish identically, i.e.

(1.3) $a)\ \mathcal{B}_k F = 0$ and $b)\ \mathcal{B}_k y^i = 0.$

The tensor H^i_{jkh} is called *h-curvature tensor*. It is positively homogeneous of degree zero in y^i and skew-symmetric in its last two lower indices which defined by

$$H^i_{jkh} := \partial_h G^i_{jk} + G^r_{jk} G^i_{rh} + G^i_{rk} G^r_j - h/k.$$

In view of Euler's theorem on homogeneous functions we have the following relations

(1.4) $a)\ \dot\partial_j H^i_{kh} = H^i_{jkh},$ $b)\ y^r \dot\partial_r H^i_{jkh} = y^r \dot\partial_j \dot\partial_r H^i_{kh} = 0,$ $c)\ H^i_{jkh} y^j = H^i_{kh},$

$d)\ H_{ijkh} := g_{jr} H^r_{ikh}$ $e)\ H^i_{kh} = \dot\partial_k H^i_h,$ $f)\ H^i_{kh} y^k = H^i_h,$ $g)\ H_{jk.h} = g_{ik} H^i_{jh},$

$h)\ H_{jk} = H^r_{jkr},$ $i)\ H^r_{rkh} = H_{kh} - H_{hk},$ $l)\ H_k = H^r_{kr},$ $m)\ H = \frac{1}{n-1} H^r_r$ and

$n)\ H_k y^k = H$

2. NORMAL PROJECTIVE CURVATURE TENSOR

K. Yano [9] defined the normal projective connection by $\Pi^i_{jk} = G^i_{jk} - \frac{1}{n+1} y^i G^r_{jkr}$.

The connection coefficients Π^i_{jk} is positively homogeneous of degree zero in y^i's and symmetric in their lower indices. The covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k in the sense of Berwald is given by

(2.1) $a)\ \mathcal{B}_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j)\Pi^r_{ks} y^s + T^r_j \Pi^i_{kr} - T^i_r \Pi^r_{kj}$.

The commutation formula for the above covariant derivative is given by

$b)\ \mathcal{B}_k \mathcal{B}_h T^i_j - \mathcal{B}_h \mathcal{B}_k T^i_j = T^r_j N^i_{rkh} - T^i_r N^r_{jkh} - \dot{\partial}_r T^i_j N^r_{skh} y^s$.

P.N. Pandey [4] obtained a relation between the normal projective curvature tensor N^i_{jkh} and Berwald curvature tenser H^i_{jkh} as follows:

(2.2) $N^i_{jkh} = H^i_{jkh} - \frac{1}{n+1} y^i \dot{\partial}_j H^r_{rkh}$.

The normal projective curvature tensor N^i_{jkh} is homogeneous of degree zero in y^i. For the tensor N^i_{jkh} we have the identities

(2.3) $a)\ N^r_{rkh} = H^r_{rkh}$, $b)\ N^i_{jkh} y^j = H^i_{kh}$, $c)\ N^i_{jkh} = -N^i_{jhk}$,

$d)\ N^i_{jkh} + N^i_{khj} + N^i_{kjh} = 0$ and $e)\ N_{jk} = N^r_{jkr}$.

3. DECOMPOSITION OF NORMAL PROJECTIVE CURVATURE TENSOR IN FINSLER SPACE

Let us consider the decomposition of the normal projective curvature tensor N^i_{jkh} of a Finsler space is of the type (1,3) as follows :

(3.1) $N^i_{jkh} = y^i Y_{\boldsymbol{jkh}}$

where Y_{jkh} is non-zero tensor filed called *decomposition tensor field*.

Further considering the decomposition of the tensor field (3.1) in the form

(3.2) $a)\ N^i_{jkh} = y^i y_j Y_{kh}$ or $b)\ N^i_{jkh} = y^i y_h Y_{jk}$.

Let us define

(3.3) $y^j \lambda_j = \sigma$,

such λ_j as recurrence vector and σ is decomposition scalar.

In view of (3.1), the identities (2.3c) and (2.3d), can be written as

(3.4) $a)\ Y_{jkh} + Y_{jhk} = 0$ and $b)\ Y_{jkh} + Y_{khj} + Y_{hjk} = 0$.

In view of (2.3a), the contraction of the indices i and j in (3.1) gives

(3.5) $H^r_{rkh} = y^r Y_{rkh}$.

Using (1.4i) in (3.5), we get

(3.6) $Y_{rkh} = \frac{1}{y^r}(H_{hk} - H_{kh})$.

In view of (3.1), equ. (3.6) can be written as

(3.7) $N^i_{jkh} = \frac{y^i}{y^j}(H_{hk} - H_{kh})$.

Transecting (3.7) by y^j and using (2.3b), we get

(3.8) $H^i_{kh} = y^i(H_{hk} - H_{kh})$.

Thus, we conclude

Theorem 3.1. *If the normal projective curvature tensor N^i_{jkh} of a Finsler space is decomposable in the form (3.1), then the normal projective curvature tensor N^i_{jkh} and the h(v)- torsion tensor H^i_{kh} are defined by (3.7) and (3.8) respectively, the tensor H^r_{rkh} is decomposable in the form (3.5) and the decomposable tensor field Y_{jkh} satisfies (3.6).*

Contraction of the indices i and j in (3.2a), using (2.3a) and (1.2), we get

(3.9) $$H^r_{rkh} = F^2 Y_{kh}.$$

Using (1.4i) in (3.9), we get

(3.10) $$Y_{kh} = \frac{1}{F^2}(H_{hk} - H_{kh}).$$

In view of (3.2a), equ. (3.10) can be written as

(3.11) $$N^i_{jkh} = y^i y_j \frac{1}{F^2}(H_{hk} - H_{kh}).$$

Transecting (3.11) by y^j and using (2.3b), we get

(3.12) $$H^i_{kh} = y^i(H_{hk} - H_{kh}).$$

Thus, we conclude

Theorem 3.2. *If the normal projective curvature tensor N^i_{jkh} of a Finsler space is decomposable in the form (3.2a), then the normal projective curvature tensor N^i_{jkh} and the h(v)- torsion tensor H^i_{kh} are defined by (3.11) and (3.12) respectively, the tensor H^r_{rkh} is decomposable in the form (3.9) and the decomposable tensor field Y_{kh} satisfies (3.10).*

4. DECOMPOSITION OF NORMAL PROJECTIVE CURVATURE TENSOR N^i_{jkh} in NPR- F_n

P. N. Pandey [4] discussed a Finsler space F_n for which the normal projective curvature tensor N^i_{jkh} satisfies the recurrence property with respect to Berwald's connection coefficients and called it *NPR-Finsler space*. Thus, *NPR*-Finsler space is characterized by

(4.1) $$\mathcal{B}_m N^i_{jkh} = \lambda_m N^i_{jkh}, \qquad N^i_{jkh} \neq 0,$$

where λ_m non-zero covariant vector field is recurrence vector field .

Note 4.1. In this paper, we shall denote such space briefly by *NPR* -F_n.

Let us consider a Finsler space whose normal projective curvature tensor N^i_{jkh} satisfies the condition (4.1).

Transvecting (4.1) by y^j, using (1.3b) and (2.3b), we get

(4.2) $$\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh}.$$

Contraction of the indices i and j in (4.1) and using (2.3a), we get

(4.3) $$\mathcal{B}_m H^r_{rkh} = \lambda_m H^r_{rkh}.$$

Transvecting (4.2) by y^k, using(1.3b) and (1.4f), we get

(4.4) $$\mathcal{B}_m H^i_h = \lambda_m H^i_h.$$

Contraction of the indices i and h in (4.2) using (1.4l), we get

(4.5) $$B_m H_k = \lambda_m H_k.$$

Transvecting (4.5) by y^k, using (1.3b) and (1.4n), we get

(4.6) $$B_m H = \lambda_m H.$$

We know that the normal projective curvature tensor N^i_{jkh} satisfies the following:

(4.7) $$\lambda_m N^i_{jkh} + \lambda_k N^i_{jhm} + \lambda_h N^i_{jmk} = 0.$$

Differentiating (3.1) covariantly with respect to x^m in the sense of Berwald, using (4.1), (3.1) and (1.3b), we get

(4.8) $\mathcal{B}_m Y_{jkh} = \lambda_m Y_{jkh}$,

where λ_m is non-zero vector filed.

Thus, we conclude

Theorem 4.1. *In an NPR- F_n , under the decomposition (3.1), the decomposition tensor field Y_{jkh} behaves like a recurrent tensor field.*

Differentiating (4.8) covariantly with respect to x^l in the sense of Berwald and using (4.8), we get

(4.9) $\mathcal{B}_l \mathcal{B}_m Y_{jkh} = (\mathcal{B}_l \lambda_m) Y_{jkh} + \lambda_m \lambda_l Y_{jkh}$.

Interchanging the indices m and l in (4.9) and subtracting the equation obtained from (4.9), we get

(4.10) $\mathcal{B}_l \mathcal{B}_m Y_{jkh} - \mathcal{B}_m \mathcal{B}_l Y_{jkh} = (\mathcal{B}_l \lambda_m - \mathcal{B}_m \lambda_l) Y_{jkh}$.

Using the commutation formula (2.1b) in (4.10), we get

(4.11) $(\mathcal{B}_l \lambda_m - \mathcal{B}_m \lambda_l) Y_{jkh} = -\big(Y_{rkh} N^r_{jml} + Y_{jrh} N^r_{kml} + Y_{jkr} N^r_{hml} + \dot{\partial}_r Y_{jkh} N^r_{sml} y^s\big)$.

In view of (3.1) and the homogeneity property of Y_{jkh}, equ. (4.11) can be written as

(4.12) $(\mathcal{B}_l \lambda_m - \mathcal{B}_m \lambda_l) Y_{jkh} = -\big(Y_{rkh} Y_{jml} + Y_{jrh} Y_{kml} + Y_{jkr} Y_{hml} - Y_{jkh} Y_{rml}\big) y^r$.

Differentiating (4.12) covariantly with respect to x^n in the sense of Berwald, using (4.8), (1.3b) and (4.12), we get

(4.13) $\mathcal{B}_n (\mathcal{B}_l \lambda_m - \mathcal{B}_m \lambda_l) = \lambda_n (\mathcal{B}_l \lambda_m - \mathcal{B}_m \lambda_l)$.

Thus, we conclude

Theorem 4.2. *In an NPR- F_n , the recurrence $\mathcal{B}_l \lambda_m - \mathcal{B}_m \lambda_l$ behaves like a recurrent tensor field under the decomposition (3.1).*

Further considering the decomposition of the tensor field Y_{jkh} in the form

(4.14) $Y_{jkh} = \lambda_j Y_{kh}$.

Differentiating (4.14) covariantly with respect to x^m in the sense of Berwald, using (4.8) and (4.14), we get

(4.15) $\lambda_m \lambda_j Y_{kh} = \big(\mathcal{B}_m \lambda_j\big) Y_{kh} + \lambda_j \mathcal{B}_m Y_{kh}$.

Transvecting (4.15) by y^j , using (3.3) and (1.3a), we get

(4.16) $\mathcal{B}_m Y_{kh} = \lambda_m Y_{kh}$.

Thus, we conclude

Theorem 4.3. *In an NPR- F_n , under the decomposition (3.1) and (4.14), the tensor field Y_{kh} behaves like a recurrent tensor field.*

From (4.12) and (4.14), we get

(4.17) $\lambda_j \{(\mathcal{B}_l \lambda_m - \mathcal{B}_m \lambda_l) Y_{kh} - Y_{ml} (\lambda_k Y_{rh} + \lambda_h Y_{kr}) y^r\} = 0$.

Using (4.14) in (3.4b), we get

(4.18) $-\lambda_j Y_{kh} = \lambda_k Y_{hj} + \lambda_h Y_{jk}$.

From (4.17), (4.18), using (3.3), the fact that the vector filed λ_m and the tensor field Y_{jkh} are non-zero, we get

(4.19) $\mathcal{B}_l \lambda_m - \mathcal{B}_m \lambda_l + \sigma Y_{ml} = 0$.

Thus, we conclude

Theorem 4.4. *In an NPR-* F_n *, under the decomposition (3.1) and (4.19), the necessary and sufficient condition that* $\mathcal{B}_l\lambda_m = \mathcal{B}_m\lambda_l$ *is that* $\sigma = 0$.

Using (3.1) in (4.7), we get

(4.20) $\lambda_m Y_{jkh} + \lambda_k Y_{jhm} + \lambda_h Y_{jkm} = 0.$

In view of (4.20) and (4.14), we get

(4.21) $\lambda_m Y_{kh} + \lambda_k Y_{hm} + \lambda_h Y_{km} = 0.$

Thus, we conclude

Theorem 4.5. *In an NPR-* F_n *, under the decomposition (3.1), the decomposition tensor fields* Y_{jkh} *and* Y_{kh} *satisfy the identities (4.20) and (4.21), respectively.*

Transvecting (3.1) by y^j , using (2.3b), (3.3) and (4.14), we get

(4.22) $H^i_{kh} = \sigma y^i Y_{kh}.$

Thus, we conclude

Theorem 4.6. *In an NPR-* F_n *, under the decompositions (3.1), (3.3) and (4.14), the h(v)-torsion tensor* H^i_{kh} *is decomposable in the form (4.22).*

Differentiating (4.22) covariantly with respect to x^m in the sense of Berwald, using (1.3b), (4.2) and (4.22), we get

(4.23) $y^i \mathcal{B}_m Y_{kh} = y^i \lambda_m Y_{kh},$

which can be written as

(4.24) $\mathcal{B}_m Y_{kh} = \lambda_m Y_{kh}.$

Thus, we conclude

Theorem 4.7. *In an NPR-* F_n *, under the decomposition (4.22), the decomposition tensor field* Y_{kh} *behaves like a recurrent tensor field .*

Contracting of the indices i and m in (4.23), we get

(4.25) $y^s \mathcal{B}_s Y_{kh} = y^s \mathcal{B}_s Y_{kh}.$

Thus, we conclude

Theorem 4.8. *In an NPR-* F_n *, under the decomposition (4.22), the directional derivatives of the decomposition tensor field* Y_{kh} *in the direction of* y^s *is proportional to* Y_{kh} *.*

Contraction of the indices i and h in (4.22) and using (1.4l), we get

(4.26) $H_k = Y_k,$

where $Y_{ki} y^i = Y_k.$

Differentiating (4.26) covariantly with respect to x^m in the sense of Berwald and using (4.5) and (4.26), we get

(4.27) $B_m Y_k = \lambda_m Y_k.$

Thus, we conclude

Theorem 4.9. *In an NPR-* F_n *, under the decomposition (4.26), the decomposition tensor fields* Y_k *behaves like a recurrent tensor field .*

Differentiating (4.11) covariantly with respect to x^l and x^n in the sense of Berwald and using (4.11), we get

(4.28) $B_n B_l B_m Y_{jkh} = \{B_n B_l \lambda_m + \lambda_n (B_l \lambda_m + \lambda_m \lambda_l) + \lambda_l B_n \lambda_m + \lambda_m B_n \lambda_l\} Y_{jkh}.$

Interchanging the indices l and n in (4.28) and subtracting the equation obtained from (4.28), we get

(4.29) $B_n B_l B_m Y_{jkh} - B_l B_n B_m Y_{jkh} = \{(B_n B_l \lambda_m - B_l B_n \lambda_m) + \lambda_m (B_n \lambda_l - B_l \lambda_n)\} Y_{jkh}$.

Using the commutation formula (2.1b), (3.1) and (4.8) in (4.29), we get

(4.30) $\lambda_m (B_n \lambda_l - B_l \lambda_n) Y_{jkh} = 0.$

Cyclically permeation of m, *l* and n in (4.30), we get

(4.31) $\lambda_m (B_n \lambda_l - B_l \lambda_n) + \lambda_l (B_m \lambda_n - B_n \lambda_m) + \lambda_n (B_m \lambda_l - B_l \lambda_m) = 0,$ where $Y_{jkh} \neq 0.$

Thus, we conclude

Theorem 4.10. *In an NPR- F_n, under the decomposition (3.1) the recurrence vector field λ_l satisfies relation (4.31).*

Using the commutation formula (2.1b), (3.1), (4.14) and (4.8) in (4.29), we get

(4.32) $B_n B_l B_m Y_{jkh} - B_l B_n B_m Y_{jkh} = \{\lambda_m (B_n \lambda_l - B_l \lambda_n) - \lambda_r Y_{mln} y^r\} Y_{jkh}$.

Cyclically permeation m, *l* and n in (4.32), using (4.31) and (3.4b), we get

(4.33) $\{B_l (B_m B_n - B_n B_m) + B_m (B_n B_l - B_l B_n) + B_n (B_l B_m - B_m B_l)\} Y_{jkh} = 0.$

Thus, we conclude

Theorem 4.11. *In an NPR- F_n, under the decomposition (3.1) the decomposition tensor field Y_{jkh} satisfies the relation (4.33).*

REFERENCES

[1] Pande, H.D. and Khan, T. A.: General Decomposition of Berwald's Curvature Tensor Fields in Recurrent Finsler Spaces, Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat. Natur., 55, (1973), 680-685.

[2] Pande, H.D. and Shukla, H. S.: On the Decomposition of Curvature Tensor Fields in Recurrent Finsler Spaces, J. of Pure and Applied Maths. Vol. 8, No. 4, (1977), 418-424.

[3] Pandey, P.N.: On Decomposability of Curvature Tensor of Finsler Manifold. Acta. Math. Acad. Sci. Hunger. 38, (1981), 109-116.

[4] Pandey, P.N.: Some Identities in an NPR-Finsler Manifold, J. of Math., (India), Vol. 24, (1982), 135-143.

[5] Prateek Mishra, Kaushal Srivastava and Mishra, S. B.: Decomposition of Curvature Tensor Field $R^{i}_{\overset{++}{\underset{++}{jkh}}}(x, y)$in a Finsler Spaces Equipped With Non-Symmetric Connection, Journal of Chemical, Biological and Physical Sciences. An Int. Peer Review E-3 J. of Sci. Vol.3, No.2, (February 2013-April 2013), 1498-1503.

[6] Ram Hit.: Decomposition of Berwald's Curvature Tensor Fields, Ann. Fac. Sci. (Kinshasa), 1, (1975), 220-226.

[7] Sinha, B. B. and Singh, S. P.: On Decomposition of Recurrent Curvature Tensor Fields in Finsler Spaces, Bull. Cal. Math. Soc. 62, (1970), 91-96.

[8] Takano, K.: Recurrent Affine Motion in S–AK_N^* Space, Tensor, N.S., 17, (1966), 321-326.

[9] Yano, K.: *The Theory of Lie-Derivatives and its Applications*, North – Holland publishing Co., (Amsterdam), (1957).